Markscheme

November 2015

Calculus

Higher level

Paper 3

12 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a valid **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2015**". It is essential that you read this document before you start marking.
In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets eg (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**).*
*A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)***.

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT***.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features (for example, TI-89) are not allowed.

Calculator notation
The Mathematics HL guide says:

Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

1. (a) consider upper or lower limits ***M1***

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1 \left(= f(0)\right),\ \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1-x) = 1\left(= f(0)\right)$ ***A1***

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ so f is continuous ***AG***

[2 marks]

(b) $\lim_{h\to 0^-} \frac{1-1}{h} = 0$ ***M1A1***

$\lim_{h\to 0^+} \frac{1-h-1}{h} = \lim_{h\to 0^+} (-1) = -1$ ***A1***

> **Note:** Award ***M1*** for an attempt to find limits in either case.

$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} \neq \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$ so f is not differentiable ***AG***

> **Note:** Award ***M1A1A0*** for correct differentiation of left and right sides, ONLY if limits then used to show that both functions have different limits.

[3 marks]

Total [5 marks]

2. (a) $f'(x) = e^x \sin x + e^x \cos x$ ***M1A1***

$f''(x) = e^x \sin x + e^x \cos x - e^x \sin x + e^x \cos x = 2e^x \cos x$ ***A1***

$= 2\left(e^x \sin x + e^x \cos x - e^x \sin x\right)$ ***M1***

$= 2\left(f'(x) - f(x)\right)$ ***AG***

[4 marks]

(b) $f(0) = 0,\ f'(0) = 1,\ f''(0) = 2(1-0) = 2$ ***(M1)A1***

> **Note:** Award ***M1*** for attempt to find $f(0)$, $f'(0)$ and $f''(0)$.

$f'''(x) = 2\left(f''(x) - f'(x)\right)$ ***(M1)***

$f'''(0) = 2(2-1) = 2,\ f^{IV}(0) = 2(2-2) = 0,\ f^{V}(0) = 2(0-2) = -4$ ***A1***

so $f(x) = x + \frac{2}{2!}x^2 + \frac{2}{3!}x^3 - \frac{4}{5!}x^5 + \ldots$ ***(M1)A1***

$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$

[6 marks]

Total [10 marks]

3. (a) if $n = 7$ then $7! > 3^7$ ***A1***

so true for $n = 7$

assume true for $n = k$ ***M1***

so $k! > 3^k$

consider $n = k + 1$

$(k+1)! = (k+1)k!$ ***M1***

$> (k+1)3^k$

$> 3.3^k$ (as $k > 6$) ***A1***

$= 3^{k+1}$

hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 7$, so true for all $n \geq 7$. ***R1***

Note: Do not award the ***R1*** if the two ***M*** marks have not been awarded.

[5 marks]

(b) consider the series $\sum_{r=7}^{\infty} a_r$ where $a_r = \frac{2^r}{r!}$ ***R1***

Note: Award the ***R1*** for starting at $r = 7$.

compare to the series $\sum_{r=7}^{\infty} b_r$ where $b_r = \frac{2^r}{3^r}$ ***M1***

$\sum_{r=7}^{\infty} b_r$ is an infinite Geometric Series with $r = \frac{2}{3}$ and hence converges ***A1***

Note: Award the ***A1*** even if series starts at $r = 1$.

as $r! > 3^r$ so $(0 <)\ a_r < b_r$ for all $r \geq 7$ ***M1R1***

as $\sum_{r=7}^{\infty} b_r$ converges and $a_r < b_r$ so $\sum_{r=7}^{\infty} a_r$ must converge

Note: Award the ***A1*** even if series starts at $r = 1$.

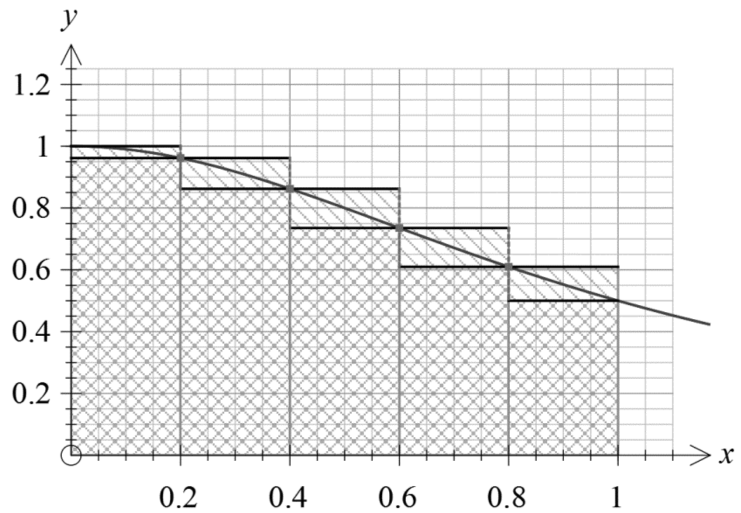
as $\sum_{r=1}^{6} a_r$ is finite, so $\sum_{r=1}^{\infty} a_r$ must converge ***R1***

Note: If the limit comparison test is used award marks to a maximum of ***R1M1A1M0A0R1***.

[6 marks]

Total [11 marks]

4. (a)



A1A1A1

A1 for upper rectangles, ***A1*** for lower rectangles, ***A1*** for curve in between with $0 \le x \le 1$

hence $\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \int_0^1 f(x)\,\mathrm{d}x < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$ ***AG***

[3 marks]

(b) attempting to integrate from 0 to 1 ***(M1)***

$\int_0^1 f(x)\,\mathrm{d}x = \left[\arctan x\right]_0^1$

$= \frac{\pi}{4}$ ***A1***

attempt to evaluate either summation ***(M1)***

$\frac{1}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \frac{\pi}{4} < \frac{1}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

hence $\frac{4}{5}\sum_{r=1}^{5} f\left(\frac{r}{5}\right) < \pi < \frac{4}{5}\sum_{r=0}^{4} f\left(\frac{r}{5}\right)$

so $2.93 < \pi < 3.33$ ***A1A1***

Note: Accept any answers that round to 2.9 and 3.3.

[5 marks]

continued…

Question 4 continued

(c) **EITHER**

recognise $\sum_{r=0}^{n-1}(-1)^r x^{2r}$ as a geometric series with $r = -x^2$ ***M1***

sum of n terms is $\frac{1-\left(-x^2\right)^n}{1--x^2} = \frac{1+(-1)^{n-1}x^{2n}}{1+x^2}$ ***M1AG***

OR

$\sum_{r=0}^{n-1}(-1)^r\left(1+x^2\right)x^{2r} = \left(1+x^2\right)x^0 - \left(1+x^2\right)x^2 + \left(1+x^2\right)x^4 + \ldots$

$+ (-1)^{n-1}\left(1+x^2\right)x^{2n-2}$ ***M1***

cancelling out middle terms ***M1***

$= 1 + (-1)^{n-1}x^{2n}$ ***AG***

[2 marks]

(d) $\sum_{r=0}^{n-1}(-1)^r x^{2r} = \frac{1}{1+x^2} + (-1)^{n-1}\frac{x^{2n}}{1+x^2}$

integrating from 0 to 1 ***M1***

$\left[\sum_{r=0}^{n-1}(-1)^r\frac{x^{2r+1}}{2r+1}\right]_0^1 = \int_0^1 f(x)\,\mathrm{d}x + (-1)^{n-1}\int_0^1\frac{x^{2n}}{1+x^2}\,\mathrm{d}x$ ***A1A1***

$\int_0^1 f(x)\,\mathrm{d}x = \frac{\pi}{4}$ ***A1***

so $\pi = 4\left(\sum_{r=0}^{n-1}\frac{(-1)^r}{2r+1} - (-1)^{n-1}\int_0^1\frac{x^{2n}}{1+x^2}\,\mathrm{d}x\right)$ ***AG***

[4 marks]

Total [14 marks]

5. (a) gradient of f at $(1, 0)$ is $1 - 0^2 = 1$ and the gradient of g at $(1, 0)$ is $0 - 1^2 = -1$ ***A1***

so gradient of normal is 1 ***A1***

= Gradient of the tangent of f at $(1,0)$ ***AG***

[2 marks]

(b) $\frac{dy}{dx} - y = -x^2$

integrating factor is $e^{\int -1\,dx} = e^{-x}$ ***M1***

$ye^{-x} = \int -x^2 e^{-x}\,dx$ ***A1***

$= x^2 e^{-x} - \int 2xe^{-x}\,dx$ ***M1***

$= x^2e^{-x} + 2xe^{-x} - \int 2e^{-x}\,dx$

$= x^2e^{-x} + 2xe^{-x} + 2e^{-x} + c$ ***A1***

$\Rightarrow g(x) = x^2 + 2x + 2 + ce^x$

$g(1) = 0 \Rightarrow c = -\frac{5}{e}$ ***M1***

$\Rightarrow g(x) = x^2 + 2x + 2 - 5e^{x-1}$ ***A1***

[6 marks]

(c) use of $y_{n+1} = y_n + hf'(x_n, y_n)$ ***(M1)***

$x_0 = 1,\ y_0 = 0$

$x_1 = 1.2,\ y_1 = 0.2$ ***A1***

$x_2 = 1.4,\ y_2 = 0.432$ ***(M1)(A1)***

$x_3 = 1.6,\ y_3 = 0.67467\ldots$

$x_4 = 1.8,\ y_4 = 0.90363\ldots$

$x_5 = 2,\ y_5 = 1.1003255...$

answer $= 1.10033$ ***A1*** ***N3***

Note: Award ***A0*** or ***N1*** if 1.10 given as answer.

[5 marks]

(d) at the point $(1, 0)$, the gradient of f is positive so the graph of f passes into the first quadrant for $x > 1$

in the first quadrant below the curve $x - y^2 = 0$ the gradient of f is positive ***R1***

the curve $x - y^2 = 0$ has positive gradient in the first quadrant ***R1***

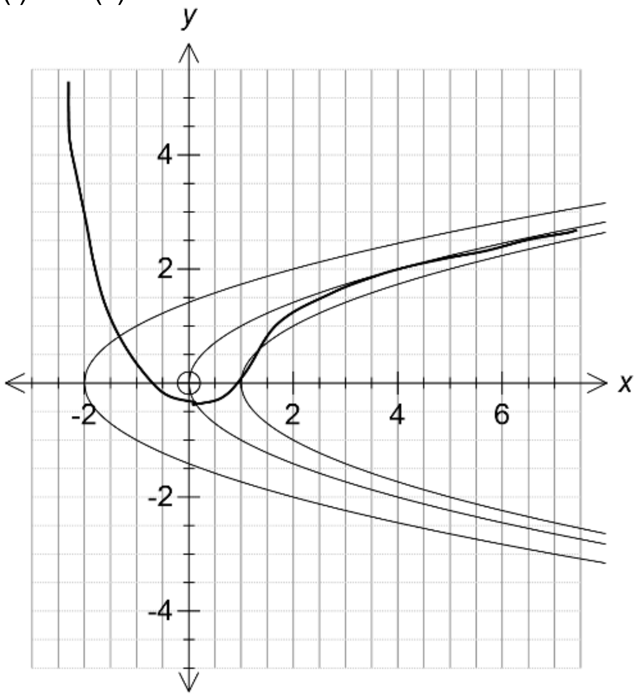
if f were to reach $x - y^2 = 0$ it would have gradient of zero, and therefore would not cross ***R1***

[3 marks]

continued…

Question 5 continued

(e) (i) and (ii)



A4

Note:	Award ***A1*** for 3 correct isoclines. Award ***A1*** for f not reaching $x - y^2 = 0$. Award ***A1*** for turning point of f on $x - y^2 = 0$. Award ***A1*** for negative gradient to the left of the turning point.

Note: Award ***A1*** for correct shape and position if curve drawn without any isoclines.

[4 marks]

Total [20 marks]